REAL VERSUS COMPLEX GAUSSIAN DISTRIBUTIONS FOR DIGITAL COMMUNICATION

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ABSTRACT*: Concept of both real and complex Gaussian Random Variables (GRVs) and their corresponding Gaussian Random processes (GRPs) is very important and critical for understanding and designing of a real Communication System. In this paper, we thus discuss and compute the various parameters involved in characterizing the real and complex Gaussian distributions completely for one and multiple Random Variables (RVs). We first carry out our analysis for one real GRV and then extend our work to multiple real GRVs. Furthermore, we apply our technique for studying one complex GRV and subsequently extend it to the analysis of multiple complex GRVs also known as multivariate complex Gaussian random vector. Finally, we make the necessary comparison between real and complex Gaussian Distributions which are the key components of Additive White Gaussian Noise (AWGN) for baseband and bandpass transmission of the digital data through non ideal channels of digital Communication Systems.*

Key Words: Complex Gaussian, *Cumulative Density Function, MGF & CF, Probability Density Function, Real Gaussian*

1 INTRODUCTION

The study of both real and complex Gaussian distributions is very essential and critical in further understanding of Gaussian Random and Stochastic Processes. The transmission of information through non-ideal channel involves addition of additive white Gaussian Noise (*AWGN*) which plays significant role in distorting the information. This noise carries the properties and characteristics either of a real or a complex Gaussian Random Process depending upon the type of transmission, *i.e.*, baseband or bandpass. The power spectral density (*PSD*) of both white and thermal noises is practically flat over a very large band (up to 1000 *GHz* at room temperature), [1–3].

Bandpass transmission contains the *PSD* of a random process to be confined to a certain passband while baseband transmission allows the *PSD* of the process to be defined around the low frequency spectrum region. Bandpass random processes can be used effectively to model modulated communication signals and bandpass noises. Like bandpass signals, we can also model the bandpass noise in terms of its in phase and quadrature components using the two famous carriers. The *PSD* of this process can easily be expressed in terms of the *PSD* of the baseband random noise process [1–5].

The Gaussian random process is perhaps the single most important random process in the area of communication. This process has a uniform *PSD* over large range of frequency spectrum and is thus known as a white Gaussian random process. The envelope of Gaussian noise process behaves like Rayleigh density distribution. This distribution can easily be fabricated from two independent real *GRVs* or one scalar complex *GRV* having zero mean and the same variance. The magnitude of scalar complex *GRV* thus follows Rayleigh density function and phase of scalar complex *GRV* follows the uniform distribution. On the other hand, when a sinusoidal signal is buried in this narrowband Gaussian noise, then the envelope of this transmission over the communication channel follows Ricean density instead of Rayleigh distribution due to non-zero mean of the noise mixed with the information. However, this density approaches to Gaussian density with certain a mean and variance for the case when amplitude of the signal is much larger than variance of the noise. The phase of the noise in this case does not follow the uniform distribution due to non-linear terms involved into its density expression [4 – 7].

This paper is organized as follows*: Section 2 computes the parameters required for complete description of Gaussian distribution for one real random variable. Extension to more than one real Gaussian random variable is carried out in section 3. Section 4 and 5 describe the detailed analysis for one and more than one complex Gaussian Random variables respectively. We make the important comparison between real and complex Gaussian Distributions in section 6. Finally, we present our conclusions in section 7.*

2 COMPUTATION OF THE PARAMETERS NEEDED FOR REAL GAUSSIAN DISTRIBUTION IN CASE OF ONE RANDOM VARIABLE

This is a very important type of continuous distribution which is extensively used in the performance analysis of many communication systems because of the addition of white Gaussian Noise (*AWGN*) in the transmitted signal when it is received by the receiver. A real Gaussian Random Variable (*RV*), $\boldsymbol{X}$ is said to be normally distributed with mean, (m_x) and variance, (σ_x^2), if its probability density function, (*PDF*) can be expressed in the following form:

$$f_X(x)=\left(\sqrt{2\pi}\sigma_x\right)^{-1} e^{-(x-m_x)^2/2\sigma_x^2} \sim N\left(m_x,\sigma_x^2\right);\ -\infty<x<\infty \qquad (1)$$

We'll soon show that the area bounded by $f_{\boldsymbol{X}}(x)$ over the entire x-axis is unity. Since linear transformation applied to Gaussian Random variables does not change its property [6], so we introduce such type of transformation as, $\boldsymbol{Y} = (\boldsymbol{X} - m_x)/\sigma_x$ in order to convert this *PDF* into standard Gaussian *PDF* [7], whose mean is zero and variance equal to unity. The *PDF* of the *RV*, $\boldsymbol{Y}$ using the results of linear transformation thus becomes

$$f_Y(y)=\left(\sqrt{2\pi}\right)^{-1} exp\left(-y^2/2\right) \sim N(0,1);\ -\infty<y<\infty \qquad (2)$$

Now, we'll first show that area bounded by this *PDF* over the entire *z*-axis is unity. This can easily be done by first doubling the area bounded in the right half plane, then making the substitution, $y = sqrt(2u)$ and finally employing the definition of *gamma* function. We now verify the mean and variance of the *RV*, $\boldsymbol{Y}$ using the known mean and variance of the *RV*, $\boldsymbol{X}$ using the transformation. $\sigma_y\, m_y = \sigma_y\, E(\boldsymbol{Y}) = E(\boldsymbol{X}) - E(m_x) = m_x - m_x = 0 \rightarrow m_y = 0$ since $\sigma_y \neq 0$. Now, $(\boldsymbol{Y} - m_y)^2 = \boldsymbol{Y}^2 = (\sigma_x)^{-2}\ x\ (\boldsymbol{X} - m_x)^2$. Thus, $Var(\boldsymbol{Y}) = (\sigma_y)^2 = (\sigma_x)^{-2}\ x\ Var(\boldsymbol{X}) = 1$, *hence verified*! This immediately shows that the mean and the variance of the *RV*, $\boldsymbol{X}$ given in *eq*. (*1*) are correct and the area bounded by its *PDF* over entire *x*-axis is also unity.

Similarly, we can also compute the cumulative distribution function (*CDF*) of the *RV*, $\boldsymbol{Y}$ first and then apply this result in obtaining the *CDF* of the *RV*, $\boldsymbol{X}$. Thus, we make use of the definition for *CDF* of the *RV*, $\boldsymbol{Y}$ as shown below:

$$F_{\boldsymbol{Y}}(y) = \boldsymbol{Pr}\{\boldsymbol{Y} \leq y\} = \int_{-\infty}^{y} f_{\boldsymbol{Y}}(u)\,du = 1 - \int_{y}^{\infty} f_{\boldsymbol{Y}}(u)\,du$$

$$= 1 - 1/\sqrt{2\pi}\int_{y}^{\infty} e^{-u^2/2}du = \boxed{1 - Q(y)} \tag{3}$$

Where $Q(y)$ is the Marcum's $\boldsymbol{Q}$-function whose value is given by the area contained in the right tail of the standard normal curve, $N(0, 1)$ starting from *y* to infinity. *CDF* of the *RV*, $\boldsymbol{X}$ using *eq*.(*3*) can be obtained utilizing its definition as

$$\begin{aligned} F_{\boldsymbol{X}}(x) &= \boldsymbol{Pr}\{\boldsymbol{X} \leq x\} \\ &= \boldsymbol{Pr}\{\boldsymbol{X} \leq x\} = \boldsymbol{Pr}\{(\sigma_x \boldsymbol{Y} + m_x) \leq x\} \\ &= \boldsymbol{Pr}\{\boldsymbol{Y} \leq (x - m_x)/\sigma_x\} = F_{\boldsymbol{Y}}(y)\big|_{y=(x-m_x)/\sigma_x} \end{aligned}$$

$$\boxed{F_{\boldsymbol{X}}(x) = 1 - Q(x - m_x)/\sigma_x} \tag{4}$$

We now make use of *eq*. (*4*) and compute the probability that the value of the *RV*, $\boldsymbol{X}$ lies within *one, two* and *three* standard deviation(s) away from the mean, *i.e.* $m - k\sigma < x < m + k\sigma$, where $k = 1, 2, 3$ respectively. Hence, we compute the following three important results of this distribution as:

$$\Rightarrow \boldsymbol{Pr}\{m - \sigma < \boldsymbol{X} \leq m + \sigma\} = F_X(m+\sigma) - F_X(m-\sigma)$$

$$= (1 - Q(1)) - (1 - Q(-1)) = (1 - Q(1)) - (1 - (1 - Q(1)))$$

$$= 1 - 2Q(1) = 1 - 2\times 0.15866 = 0.6827 = \boxed{68.27\%}$$

$$\Rightarrow \boldsymbol{Pr}\{m - 2\sigma < \boldsymbol{X} \leq m + 2\sigma\} = F_X(m+2\sigma) - F_X(m-2\sigma)$$

$$= 1 - 2Q(2) = 1 - 2\times 0.02275 = 0.9445 = \boxed{94.45\%}$$

$$\Rightarrow \boldsymbol{Pr}\{m - 3\sigma < \boldsymbol{X} \leq m + 3\sigma\} = F_X(m+3\sigma) - F_X(m-3\sigma)$$

$$= 1 - 2Q(3) = 1 - 2\times 0.00135 = 0.9973 = \boxed{99.73\%} \tag{5}$$

We now proceed onwards to compute the moments of the *RV*, $\boldsymbol{X}$ from the knowledge of the moments of the *RV*, $\boldsymbol{Y}$. For odd values of $n = 2k + 1$, where $k\ \varepsilon\ Z^+$, we immediately conclude that all odd moments of the *RV*, $\boldsymbol{Y}$ are zero. This is due to the fact that the integrand $(y^n f_{\boldsymbol{Y}}(y))$ becomes an *odd* function and from Calculus, carrying out integration of an odd function within the interval having origin as an intermediate value results into a zero value. Thus, we show

$$E(\boldsymbol{Y}^n) = E((\boldsymbol{X} - m_x)/\sigma_x)^n = 0$$

$$\boxed{E(\boldsymbol{X} - m_x)^n = 0} = 0 \times \sigma_x^n;\ \ for\ \ n = 2k + 1 \tag{6}$$

For even values of $n = 2k$, we compute the moments of $\boldsymbol{Y}$ as

$$E[\boldsymbol{Y}^n] = \int_{-\infty}^{\infty} y^n f_{\boldsymbol{Y}}(y)\,dy = \int_{-\infty}^{\infty} \underbrace{\underbrace{y^{2k}}_{even}\ \underbrace{f_{\boldsymbol{Y}}(y)}_{even}}_{even}\,dy = 2\int_0^{\infty} (y^2)^k f_{\boldsymbol{Y}}(y)\,dy$$

$$= \sqrt{2/\pi}\int_0^{\infty}(y^2)^k e^{-y^2/2}dy = \sqrt{2/\pi}\int_0^{\infty}(2u)^k e^{-u}\,du/\sqrt{2u}$$

$$= 2^k/\sqrt{\pi}\int_0^{\infty}(u)^{k-1/2}e^{-u}du = \boxed{2^k/\sqrt{\pi}\,\Gamma(k+1/2)}$$

$$\Gamma(k+1/2) = \Gamma((k-1/2)+1) = (k-1/2)\Gamma(k-1/2)$$

$$= \frac{(2k-1)(2k-3)(2k-5)\ldots 3.1}{(2.2.2\ldots 2.2) = 2^k}\Gamma(1/2)$$

$$\frac{2^k}{\sqrt{\pi}}\Gamma(k+1/2) = (2k-1)(2k-3)(2k-5)\ldots 3.1$$

$$= \frac{2k(2k-1)(2k-2)(2k-3)(2k-4)(2k-5)(2k-6)\ldots 4.3.2.1}{2k(2k-2)(2k-4)(2k-6)\ldots 6.4.2}$$

$$Thus,\ E(\boldsymbol{Y}^n) = \frac{(2k)!}{(2)^k.(k!)}$$

$$Hence,\ \boxed{E(\boldsymbol{X} - m_x)^n = \frac{(2k)!}{(2)^k.(k!)}\sigma_x^{2k}};\ \ for\ \ n = 2k \tag{7}$$

We next move on to the moment generating function (*MGF*) and characteristic function (*CF*) of the *RV*, $\boldsymbol{X}$. Again, we'll follow an indirect approach, *i.e*., we will first compute the *MGF* and *CF* of the *RV*, $\boldsymbol{Y}$ and then will compute the desired *MGF* and *CF* of the *RV*, $\boldsymbol{X}$ by using these results. Thus, we apply the definition of *MGF* to the *RV*, $\boldsymbol{Y}$ and proceed as

$$M_{\boldsymbol{Y}}(s) = E\left[e^{s\boldsymbol{Y}}\right] = \int_{-\infty}^{\infty} e^{sy} f_{\boldsymbol{Y}}(y)\,dy = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{sy}e^{-y^2/2}dy$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-(y^2-2sy)/2}dy = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-(y^2-2sy+s^2-s^2)/2}dy$$

$$= e^{s^2/2} \times \underbrace{\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-(y-s)^2/2}dy}_{\sim N(s,1)=1} = \boxed{e^{s^2/2}}$$

$$M_{\boldsymbol{Y}}(\sigma_x s) = M_{\boldsymbol{Y}}(s)\big|_{s=\sigma_x s} = e^{\sigma_x^2 s^2/2};$$

$$M_{\boldsymbol{X}}(s) = E\left[e^{s\boldsymbol{X}}\right] = E\left[e^{s(\sigma_x \boldsymbol{X} + m_x)}\right] = e^{m_x s} \times M_{\boldsymbol{Y}}(\sigma_x s)$$

$$= e^{m_x s} \times e^{\sigma_x^2 s^2/2} = \boxed{e^{m_x s + \sigma_x^2 s^2/2}}$$

$$\phi_X(w) = M_{\boldsymbol{X}}(s)\big|_{s=jw} = \boxed{e^{jm_x w - \sigma_x^2 w^2/2}} \tag{8}$$

All moments of the *RV* $\boldsymbol{X}$ can also be generated using the moment generating function (*MGF*) approach as follows:

$$M_X(s) = E\left[e^{sX}\right]$$

$$Diff.\ w.r.t\ s$$

$$M_X^{/}(s) = \frac{d}{ds}\left(E\left[e^{sX}\right]\right) = E\left(\frac{d}{ds}\left[e^{sX}\right]\right) = E\left[Xe^{sX}\right]$$

$$M_X^{//}(s) = E\left[X^2 e^{sX}\right]$$

$$\vdots$$

$$M_X^k(s) = E\left[X^k e^{sX}\right] \Rightarrow \boxed{E\left[X^k\right] = M_X^k(s)\big|_{s=0}} \tag{9}$$

3 COMPUTATION OF THE PARAMETERS NEEDED FOR REAL GAUSSIAN DISTRIBUTION IN CASE OF TWO AND MORE RANDOM VARIABLES

Consider $\boldsymbol{X}$ and $\boldsymbol{Y}$ are two real *GRVs*. We can define their *Joint* or *Bivariate PDF*, $f_{\boldsymbol{X,Y}}(x,y)$ in terms of their five known parameters, *i.e.*, means (m_x, m_y), variances (σ^2_x, σ^2_y) of the *RVs* $\boldsymbol{X}$ and $\boldsymbol{Y}$ respectively and their correlation coefficient, ρ_{xy}. The *joint PDF* of real *GRVs* is given by [1] – [7]:

$$f_{\boldsymbol{XY}}(x,y) = A exp\left(-0.5K\left(a^2(x) - 2\rho_{xy}a(x)b(y) + b^2(y)\right)\right) \qquad (9)$$

$$where\ A = \sqrt{K}/2\pi;\ K = \left(\sigma_x\sigma_y\sqrt{1-\rho_{xy}^2}\right)^{-2}$$

$$a(x) = \sigma_y(x - m_x);\ and\ b(y) = \sigma_x(y - m_y) \qquad (10)$$

We can easily show that area bounded by the joint *PDF* over the entire x and y axes is equal to unity. This can be done as

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{XY}(x,y)\,dxdy = \int_{y=-\infty}^{\infty}\left(\underbrace{\int_{x=-\infty}^{\infty} f_{XY}(x,y)\,dx}_{f_Y(y)}\right)dy = \int_{y=-\infty}^{\infty} f_Y(y)\,dy = \boxed{1}$$

We now want to extend the representation given in *eq*. (*9*) for multivariate *GRVs* more than two. For this purpose, we need to express this representation into some suitable and convenient form which can facilitate us for further analysis. Let us express the argument of exponential function into some standard and the desirable form as explained below:

$$K\left(a^2(x) - 2\rho_{xy}a(x)b(y) + b^2(x)\right)$$

$$= K\times\left[a(x)\times a(x) - \rho_{xy}a(x)b(y) - \rho_{xy}a(x)b(y) + b(y)\times b(y)\right]$$

$$LHS = K\times\left\{\begin{bmatrix} a(x) & b(y)\end{bmatrix}_{(1\times2)}\begin{pmatrix}\sigma_y^2 & -\rho_{xy}\sigma_x\sigma_y \\ -\rho_{xy}\sigma_x\sigma_y & \sigma_x^2\end{pmatrix}_{(2\times2)}\begin{bmatrix}a(x)\\ b(y)\end{bmatrix}_{(2\times1)}\right\}$$

$$= \begin{bmatrix} a(x) & b(y)\end{bmatrix}_{(1\times2)}\underbrace{K\times\begin{pmatrix}\sigma_y^2 & -\rho_{xy}\sigma_x\sigma_y \\ -\rho_{xy}\sigma_x\sigma_y & \sigma_x^2\end{pmatrix}_{(2\times2)}}_{C_X^{-1}(say)}\begin{bmatrix}a(x)\\ b(y)\end{bmatrix}_{(2\times1)}$$

$$LHS = \left\{(\boldsymbol{X}-\boldsymbol{m})^t C_X^{-1}(\boldsymbol{X}-\boldsymbol{m})\right\},\ where$$

$$\boldsymbol{X} = \begin{pmatrix}X & Y\end{pmatrix}^t\ and\ \boldsymbol{m} = \begin{pmatrix}m_x & m_y\end{pmatrix}^t$$

$$C_{\boldsymbol{X}} = \begin{pmatrix}\sigma_x^2 & \rho_{xy}\sigma_x\sigma_y \\ \rho_{xy}\sigma_x\sigma_y & \sigma_y^2\end{pmatrix} = \begin{pmatrix}\sigma_x^2 & Cov(\boldsymbol{X},\boldsymbol{Y}) \\ Cov(\boldsymbol{Y},\boldsymbol{X}) & \sigma_y^2\end{pmatrix}_{(2\times2)(Symmetric)}$$

$$\boldsymbol{X}-\boldsymbol{m} = \begin{pmatrix}X-m_x & Y-m_y\end{pmatrix}^t\ and\ (\boldsymbol{X}-\boldsymbol{m}_X)^t = \begin{pmatrix}X-m_x & Y-m_y\end{pmatrix}$$

$$det(C_{\boldsymbol{X}}) = \sigma_x^2\sigma_y^2 - \rho_{xy}^2\sigma_x^2\sigma_y^2 = \sigma_x^2\sigma_y^2\left(1-\rho_{xy}^2\right)$$

$$\left|det(C_{\boldsymbol{X}})\right|^{1/2} = \sigma_x\sigma_y\sqrt{\left(1-\rho_{xy}^2\right)} = \left(\sqrt{K}\right)^{-1} = (2\pi A)^{-1}$$

We thus can write *eq*. (*9*) into our desired form as

$$f_{\boldsymbol{X}}(\boldsymbol{x}) = \left\{(2\pi)^{n/2}\left|det(C_{\boldsymbol{X}})\right|^{1/2}\right\}^{-1} exp\left(-0.5\left\{(\boldsymbol{X}-\boldsymbol{m}_X)^t C_X^{-1}(\boldsymbol{X}-\boldsymbol{m}_X)\right\}\right)\Big|_{for\, n=2} \qquad (11)$$

The representation of *eq*. (*11*) can easily be extended to more than two *GRVs*. Hence we conclude that multivariate real Gaussian Distribution is completely described by its two important parameters; *i*) ***mean*** vector, $m_{\boldsymbol{X}}$ and *ii*) the ***covariance*** matrix, $C_{\boldsymbol{X}}$. If the *RVs*, $\boldsymbol{X}$ and $\boldsymbol{Y}$ are uncorrelated, *i.e.*, $\rho_{xy} = 0$, then it is obvious from above that off diagonal terms of the covariance matrix become equal to zero and it reduces to just a diagonal matrix whose determinant equals to the product of all its diagonal terms which are the variances of the *GRVs* involved. The joint *PDF* in this case reduces to the product of marginal *PDFs* and makes the *RVs* $\boldsymbol{X}$ and $\boldsymbol{Y}$ independent for the case of *GRVs* only. However, this is not the case in general for other types of *RVs*.

The marginal *PDFs* of the *RVs* $\boldsymbol{X}$ and $\boldsymbol{Y}$ can easily be obtained by first expressing the joint *PDF* given in *eq*. (*9*) into product of marginal *PDF* and conditional *PDF* and then integrating out the conditional *PDF* which reduces to unity from applying the fact that it is a *PDF*. We thus conclude that marginal *PDFs* of the *RVs* $\boldsymbol{X}$ and $\boldsymbol{Y}$ are also Gaussian with their respective means and variances. However, the converse is not true in general [6]. Similarly, we can also show that conditional *PDF* (obtained by taking the ratio of joint *PDF* and the marginal *PDF* and utilizing some manipulation and simplification) is also Gaussian with its own conditional mean and conditional variance.

Joint moment generating (*MGF*) and joint characteristic functions (*CF*) can also be obtained easily by utilizing the property of linear transformation applied to *GRVs*. Thus, we proceed in the following manner:

$$\boldsymbol{W} = s_1\boldsymbol{X} + s_2\boldsymbol{Y} \sim N\left(m_w, \sigma_w^2\right)$$

$$m_w = E(\boldsymbol{W}) = E(s_1\boldsymbol{X} + s_2\boldsymbol{Y}) = s_1m_x + s_2m_y = \boldsymbol{s}^t\boldsymbol{m}_{\boldsymbol{X}}$$

$$(\boldsymbol{W} - m_w) = s_1(\boldsymbol{X} - m_x) + s_2(\boldsymbol{Y} - m_y)$$

$$(\boldsymbol{W} - m_w)^2 = s_1^2(\boldsymbol{X} - m_x)^2 + s_2^2(\boldsymbol{Y} - m_y)^2 + 2s_1s_2(\boldsymbol{X} - m_x)(\boldsymbol{Y} - m_y)$$

$$\sigma_w^2 = E(\boldsymbol{W} - m_w)^2 = s_1^2\sigma_x^2 + s_2^2\sigma_y^2 + 2s_1s_2Cov(\boldsymbol{X},\boldsymbol{Y}) = \boldsymbol{s}^tC_X\boldsymbol{s}$$

$$M_W(s) = E\left(e^{s\boldsymbol{W}}\right) = E\left(e^{s(s_1\boldsymbol{X}+s_2\boldsymbol{Y})}\right) = E\left(e^{(ss_1)\boldsymbol{X}+(ss_2)\boldsymbol{Y}}\right) = M_{\boldsymbol{XY}}(ss_1, ss_2)$$

$$M_W(s)\big|_{s=1} = M_W(1) = M_{XY}(ss_1, ss_2)\big|_{s=1} = M_{\boldsymbol{XY}}(s_1, s_2)$$

$$exp\left(m_w + 0.5\sigma_w^2\right) = M_{\boldsymbol{XY}}(s_1, s_2)$$

$$\boxed{\begin{array}{l} exp\left(\boldsymbol{s}^t\boldsymbol{m} + 0.5\boldsymbol{s}^tC_X\boldsymbol{s}\right) = M_{\boldsymbol{XY}}(s_1,s_2) \\ \phi_{XY}(w_1,w_2) = M_{\boldsymbol{XY}}(s_1,s_2)\big|_{s_i=jw_i};\ for\ i=1,2\end{array}};where\ \boldsymbol{s} = j\boldsymbol{w} = \begin{pmatrix}s_1 & s_2\end{pmatrix}^t \qquad (12)$$

The joint moments of the *GRVs* can also easily be obtained by extending the definition and the result of *eq*. (*9*) to more than one *RV* as [6]

$$E\left[\boldsymbol{X}^i\boldsymbol{Y}^j\right] = \frac{\partial^{i+j}}{\partial s_2^j\partial s_1^i}\left(M_{\boldsymbol{XY}}(s_1,s_2)\right)\Big|_{\substack{s_1=0;\\ s_2=0}} \qquad (13)$$

where i and j both belong to the set of positive integers.

4. COMPUTATION OF THE PARAMETERS NEEDED FOR COMPLEX GAUSSIAN DISTRIBUTION IN CASE OF ONE RANDOM VARIABLE

A complex Gaussian *RV*, $\boldsymbol{Z}$ is defined to be $\boldsymbol{Z} = \boldsymbol{X} + j\boldsymbol{Y}$, where $\boldsymbol{X}$ and $\boldsymbol{Y}$ are both real *GRVs*. The complex Gaussian *PDF* arises naturally from a consideration of the distribution of the complex envelope of a bandpass process. We can describe the *PDF* of scalar complex *RV*, $\boldsymbol{Z}$ by assuming the *RV* $\boldsymbol{X}$ to be independent from the *RV* $\boldsymbol{Y}$. Both $\boldsymbol{X}$ and $\boldsymbol{Y}$ are *GRVs* distributed as $N(m_x, \sigma^2)$ and $N(m_y, \sigma^2)$. Mathematical description of the *PDF* of $\boldsymbol{Z}$ is achieved by computing first the joint *PDF* of the *RVs* $\boldsymbol{X}$ and $\boldsymbol{Y}$ as [6] – [7]

$$f_{X,Y}(x,y) = f_{\boldsymbol{X}}(x)\times f_{\boldsymbol{Y}}(y) = \boldsymbol{N}\left(m_x,\sigma^2\right)\times\boldsymbol{N}\left(m_y,\sigma^2\right)$$

$$= \left(\sqrt{2\pi}\sigma\right)^{-1} exp\left(-(x-m_x)^2/2\sigma^2\right)\times\left(\sqrt{2\pi}\sigma\right)^{-1} exp\left(-(y-m_y)^2/2\sigma^2\right)$$

$$= \left(\pi\left(2\sigma^2\right)\right)^{-1} exp\left(-\left\{(x-m_x)^2 + (y-m_y)^2\right\}/2\sigma^2\right);$$

$$= \left\{(2\pi)^{n/2}\left|det(C_{\boldsymbol{X}})\right|^{1/2}\right\}^{-1} exp\left(-0.5\left\{(\boldsymbol{X}-\boldsymbol{m})^t C_X^{-1}(\boldsymbol{X}-\boldsymbol{m})\right\}\right)\Big|_{n=2} \qquad (14)$$

$$\boldsymbol{Z} = \boldsymbol{X} + j\boldsymbol{Y};\ m_Z = E(\boldsymbol{X} + j\boldsymbol{Y}) = E(\boldsymbol{X}) + jE(\boldsymbol{Y}) = m_x + jm_y;$$

$$(\boldsymbol{Z} - m_Z) = (\boldsymbol{X} - m_x) + j(\boldsymbol{Y} - m_y);$$

$$Var(\boldsymbol{Z}) = \sigma_Z^2 = E\left|(\boldsymbol{Z} - m_Z)\right|^2 = E\left\{(\boldsymbol{X} - m_x)^2 + (\boldsymbol{Y} - m_y)^2\right\} = 2\sigma^2;$$

$$Thus, \boxed{f_Z(z) = \left(\pi\sigma_Z^2\right)^{-1} exp\left\{-\left(\left|(\boldsymbol{Z} - m_Z)\right|^2/\sigma_z^2\right)\right\}} \qquad (15)$$

Comparison of *eq*. (*15*) with *eq*. (*14*) reveals us that $Var(\boldsymbol{Z})$, which is equal to covariance matrix, C_Z for $n = 1$ and $|det(C_Z)| = 2^n\ |det(C_X)|^{1/2}$ and $(\boldsymbol{X} - \boldsymbol{m})^t\ (C_X)^{-1}(\boldsymbol{X} - \boldsymbol{m}) = 2\ x\ (\boldsymbol{Z} - m_Z)^H\ (C_Z)^{-1}(\boldsymbol{Z} - m_Z)$, where H denote Hermitian transpose of the vector which contains transpose and conjugation operations jointly. Thus, *PDF* of $\boldsymbol{Z}$ computed in *eq*. (*15*) can also be written in most suitable form as

$$f_Z(z) = \left(\pi^n \left|det(C_Z)\right|\right)^{-1} exp\left\{-\left(\boldsymbol{Z} - m_Z\right)^H C_Z^{-1}\left(\boldsymbol{Z} - m_Z\right)\right\}\Big|_{n=1} = \boldsymbol{CN}\left(m_Z, \sigma_Z^2\right) \quad (16)$$

The representation in *eq.* (*16*) is called the complex Gaussian *PDF* for a scalar *RV*, $\boldsymbol{Z}$ and is denoted by $CN\left(m_Z, \sigma_Z^2\right)$.

5. COMPUTATION OF THE PARAMETERS NEEDED FOR COMPLEX GAUSSIAN RANDOM VECTOR

We now consider a complex random vector, $\boldsymbol{Z} = [\boldsymbol{Z}_1\ \boldsymbol{Z}_2 \ldots \boldsymbol{Z}_n]^t$, where each $\boldsymbol{Z}_i = \boldsymbol{X}_i + j\boldsymbol{Y}_i$ is distributed as The frequency response of this pulse is defined as $CN\left(m_i, \sigma_i^2\right)$ for $i = 1, 2, \ldots, n$ and is also independent. By independence, we mean that the real random vectors $(\boldsymbol{X}_1\ \ \boldsymbol{Y}_1)^t$, $(\boldsymbol{X}_2\ \ \boldsymbol{Y}_2)^t, \ldots, (\boldsymbol{X}_n\ \ \boldsymbol{Y}_n)^t$ are independent. Then, the multivariate complex Gaussian *PDF* of the random vector, $\boldsymbol{Z}$ is just the product of the marginal *PDFs* expressed in *eq.* (*15*) as

$$\begin{aligned}
f_Z(z) &= \prod_{i=1}^{n} f_{Z_i}(z_i) \\
&= \frac{1}{\left(\pi\sigma_1^2\right)\left(\pi\sigma_2^2\right)\ldots\left(\pi\sigma_n^2\right)} exp\left[-\sum_{i=1}^{n}\frac{\left|Z_i - m_i\right|^2}{\sigma_i^2}\right]; \\
&= \frac{1}{\pi^n\left(\prod_{i=1}^{n}\sigma_i^2\right)} exp\left[-\sum_{i=1}^{n}\frac{\left|Z_i - m_i\right|^2}{\sigma_i^2}\right]; \\
&= \frac{1}{\pi^n det\left(C_Z\right)} exp\left[-\left(\boldsymbol{Z} - m_Z\right)^H C_Z^{-1}\left(\boldsymbol{Z} - m_Z\right)\right] = \boldsymbol{CN}\left(\boldsymbol{m}_Z, C_Z\right) \quad (17)
\end{aligned}$$

$$where\ C_Z = E\left[\left(\boldsymbol{Z} - m_Z\right)\left(\boldsymbol{Z} - m_Z\right)^H\right] = diag\left(\sigma_1^2, \sigma_2^2, \ldots, \sigma_n^2\right);$$

The representation in *eq.* (*17*) is the multivariate complex Gaussian *PDF* and it is denoted by $CN\left(\boldsymbol{m}_Z, C_Z\right)$. It has been derived for the case of independent complex *GRVs* but however it is also valid for correlated complex *GRVs* [1], [6] – [7]. The proof is not a difficult one but we are able to define proper and circularly symmetric Gaussian random vectors as $C_X = C_Y$ (*symmetric*) and $C_{XY} = -C_{YX}$ (*skew symmetric*). Also $C_Z = C_X + C_Y + j(C_{YX} - C_{XY}) = 2C_X + j2C_{YX}$; $C_X = C_Y = ½\ Re(C_Z)$ and $C_{YX} = -C_{XY} = ½\ Im(C_Z)$ and

$$C_Z = \begin{pmatrix} C_X & C_{XY} \\ -C_{XY} & C_X \end{pmatrix} \quad (18)$$

where the matrices C_X and C_Y are the covariance matrices of real random vectors $\boldsymbol{X}$ and $\boldsymbol{Y}$ respectively and hence they are symmetric and nonnegative definite. Similarly C_{XY} denote the correlation matrix among the real random vectors $\boldsymbol{X}$ and $\boldsymbol{Y}$ and it is obvious that $(C_{XY})^t = C_{YX}$ as it is evident from their definitions [7]. In a similar manner, we can also express the real multivariate Gaussian *PDF* of the random vector, $\boldsymbol{X} = [X_1\ X_2\ \ldots\ X_n\quad Y_1\ Y_2 \ldots Y_n]^t$ using *eq*. (*11*) as

$$\begin{aligned}
f_X(\boldsymbol{x}) &= \left\{(2\pi)^{N/2}\left|det\left(C_X\right)\right|^{1/2}\right\}^{-1} exp\left(-0.5\left\{\left(\boldsymbol{X} - \boldsymbol{m}\right)^t C_X^{-1}\left(\boldsymbol{X} - \boldsymbol{m}\right)\right\}\right)\Big|_{for\ N=2n} \\
&= \left\{\pi^n\left(2^n\left|det\left(C_X\right)\right|^{1/2}\right)\right\}^{-1} exp\left(-0.5\left\{\left(\boldsymbol{X} - \boldsymbol{m}\right)^t C_X^{-1}\left(\boldsymbol{X} - \boldsymbol{m}\right)\right\}\right) \quad (19)
\end{aligned}$$

This reduces to multivariate complex Gaussian distribution of *eq*. (*16*) using the relations already explained in section 4. Thus, we conclude that in context to multivariate real Gaussian *PDF*, a multivariate complex Gaussian distribution also requires *two* important parameters for its complete description; *i.e.*, *i)* ***mean*** of the random vector = $\boldsymbol{m}_Z = \boldsymbol{m}_X + j\boldsymbol{m}_Y$ and *ii)* ***covariance*** matrix of the random vector = $C_Z = 2(C_X + jC_Y)$ whose *PDF* is given by *eq*. (*17*).

6 MOMENT GENERATING AND CHARACTERISTIC FUNCTIONS OF THE COMPLEX GAUSSIAN RANDOM VECTOR

In this section, we shall make use of the result already derived in *eq*. (*12*) along with the relations mentioned in section *4* for the case of multivariate complex Gaussian distribution. Knowing the fact that for multivariate real Gaussian vector, $\boldsymbol{X} \sim \boldsymbol{N}\left(\boldsymbol{m}, C_X\right)$, the *MGF* of $\boldsymbol{X}$ may be defined using inner product as

$$\begin{aligned}
M_X(\boldsymbol{s}) &= E\left[exp\left(\boldsymbol{s}^t\boldsymbol{x}\right)\right] \\
&= exp\left[\boldsymbol{s}^t\boldsymbol{m} + 0.5\boldsymbol{s}^t C_X \boldsymbol{s}\right] \quad (20)
\end{aligned}$$

where $\boldsymbol{s} = [s_1\ \ s_2\ \ s_3\ \ldots\ s_{2n}]^t$. Now, $\boldsymbol{x} = [\boldsymbol{X}^t\quad \boldsymbol{Y}^t]^t$ and C_X has special form, then letting $\boldsymbol{Z} = \boldsymbol{X} + j\boldsymbol{Y}$ and $m_Z = E(\boldsymbol{Z}) = E(\boldsymbol{X}) + jE(\boldsymbol{Y}) = m_X + jm_Y$ with $\boldsymbol{s} = [(s_R)^t\ \ (s_I)^t]^t$ so that $\tilde{\boldsymbol{s}} = s_R + js_I$, we can compute the result for inner product mentioned in *eq*.(*20*) as

$$\begin{aligned}
\boldsymbol{s}^t\boldsymbol{m} &= \begin{bmatrix} s_R^t & s_I^t \end{bmatrix}\begin{bmatrix} m_X \\ m_Y \end{bmatrix} = s_R^t m_X + s_I^t m_Y; \\
&= Re\left\{\left(\tilde{\boldsymbol{s}}\right)^H m\right\}; where \left(\tilde{\boldsymbol{s}}\right)^H = \left(s_R^t - js_I^t\right) \quad (21)
\end{aligned}$$

Moreover, we have mentioned in section *4* that $2s^t C_X s = \left(\tilde{s}\right)^H C_Z \tilde{s}$ and this allows us to express *MGF* and *CF* of the random vector, $\boldsymbol{Z}$ using the result of *eqs*. (*20*) & (*21*) as

$$\begin{aligned}
M_Z\left(\tilde{\boldsymbol{s}}\right) &= exp\left[Re\left\{\left(\tilde{\boldsymbol{s}}\right)^H \boldsymbol{m}\right\} + 0.25\left(\tilde{\boldsymbol{s}}\right)^H C_Z \tilde{\boldsymbol{s}}\right] \\
\phi_Z\left(\tilde{\boldsymbol{w}}\right) &= M_Z\left(\tilde{\boldsymbol{s}}\right)\Big|_{\tilde{\boldsymbol{s}} = j\tilde{\boldsymbol{w}}} \\
&= exp\left[jRe\left\{\left(\tilde{\boldsymbol{w}}\right)^H \boldsymbol{m}\right\} - 0.25\left(\tilde{\boldsymbol{w}}\right)^H C_Z \tilde{\boldsymbol{w}}\right] \quad (22)
\end{aligned}$$

Similarly, a linear transformation applied to multivariate complex Gaussian random vector, $\boldsymbol{Z}$ like multivariate real Gaussian random vector, $\boldsymbol{X}$ does not change its property. Thus, the complex random vector, $\boldsymbol{W} = A\boldsymbol{Z} + b$ will also be a multivariate complex Gaussian distribution with complex mean vector, m_W and covariance matrix, $C_W = A\ C_Z\ A^H$.

Hence, we write the complex Gaussian random vector, $\boldsymbol{W}$ as $\boldsymbol{W} \sim \boldsymbol{CN}\left(m_W, C_W\right) \sim \boldsymbol{CN}\left(Am_Z + b, AC_Z A^H\right)$.

7 COMPARISON OF REAL VS COMPLEX GAUSSIAN DISTRIBUTIONS

Real Gaussian distribution, $\boldsymbol{X}$ is normally used for baseband transmission while complex Gaussian distribution, $\boldsymbol{Z}$ is used for bandpass transmission in digital communication. Mean of $\boldsymbol{X}$ is a real quantity while it is complex for the case of $\boldsymbol{Z}$. Likewise, variance of $\boldsymbol{Z}$ is computed as $E(|\boldsymbol{Z} - m_{\boldsymbol{Z}}|)^2$ in order to make it real in comparison to the variance of $\boldsymbol{X}$ which is simply equal to $E(\boldsymbol{X} - m_{\boldsymbol{X}})^2$ and is always real. For a scalar complex Distribution, its covariance matrix, $C_{\boldsymbol{Z}} = Var(\boldsymbol{Z}) = 2Var(\boldsymbol{X}) = 2Var(\boldsymbol{Y})$ and its *PDF*, $f_{\boldsymbol{Z}}(z)$ is obtained by substituting $n = 2$ in *eq.* (*11*) along with using the relations already established in section 4. For multivariate complex random vector $\boldsymbol{Z}$, we already mentioned that $C_{\boldsymbol{Z}} = 2C_{\boldsymbol{X}} = 2C_{\boldsymbol{Y}}$ and $C_{\boldsymbol{XY}} = -C_{\boldsymbol{XY}}$ for proper and circular Gaussian *RVs* [1], [7]. This relation leads to $det^2(C_{\boldsymbol{Z}}) = 2^{2n}det(C_{\boldsymbol{X}})$ and $(\boldsymbol{X} - m_{\boldsymbol{X}})^t(C_{\boldsymbol{X}})^{-1}\ (\boldsymbol{X} - m_{\boldsymbol{X}}) = 2(\boldsymbol{Z} - m_{\boldsymbol{Z}})^H(C_{\boldsymbol{Z}})^{-1}(\boldsymbol{Z} - m_{\boldsymbol{Z}})$ in this case. Linear transformation applied to multivariate complex Gaussian vector, $\boldsymbol{Z}$ modifies the covariance matrix from $AC_{\boldsymbol{X}}A^t$ in case of multivariate real Gaussian vector, $\boldsymbol{X}$ to $AC_{\boldsymbol{Z}}A^H$. Mean vector also becomes complex due to complex nature of the random vector, $\boldsymbol{Z}$. While computing *MGF* and *CF* in case of multivariate complex Gaussian distribution, the inner product term (i.e., *first*) defined in *eq*. (*20*), and the 2nd term (*i.e.*, quadratic form) of the same equation have to be modified accordingly (*i.e.*, as per *eq*. (*22*)). Multivariate complex Gaussian distribution is quite general in nature since multivariate real Gaussian distribution expressed in *eq.* (*11*) can easily be obtained from *eq.* (*17*) by setting all $\boldsymbol{Y}$ Gaussian random variables of $\boldsymbol{Z}$ equal to zero and changing the hermitian operation to transpose only.

8 CONCULUSIONS

In this paper, we derived the various important parameters required for complete description and characterization of real and complex Gaussian distributions in case of one and multiple random variables. Their thorough understanding further helps us in the study and analysis of white and thermal noise Gaussian processes. Real multivariate Gaussian distribution is basically used for baseband transmission for example *B-PSK*, *B-FSK* and *on-off* signaling of the digital communication system. However, complex Gaussian distribution is primarily used for bandpass transmission such as *M-ary PSK* and *M-ary QAM* constellations of digital communication systems. The analysis and knowledge of these distributions is quite essential and critical in the design of receivers of digital communication systems properly. Finally, we have made comparison between multivariate real and complex Gaussian random vectors and their corresponding *PDFs* for thorough understanding and benefit of concerned readers and students of graduate programs studying at various universities of Pakistan.

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